

A THEOREM À LA FATOU FOR THE SQUARE ROOT OF POISSON KERNEL IN δ -HYPERBOLIC SPACES AND DECAY OF MATRIX COEFFICIENTS OF BOUNDARY REPRESENTATIONS

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ABSTRACT. In this note, we prove a theorem à la Fatou for the square root of Poisson Kernel in the context of quasi-convex cocompact discrete groups of isometries of δ -hyperbolic spaces. As a corollary we show that some matrix coefficients of boundary representations cannot satisfy *the weak inequality of Harish-Chandra*. Nevertheless, such matrix coefficients satisfy an inequality which can be viewed as a particular case of the inequality coming from *property RD* for boundary representations. The inequality established in this paper is based on a uniform bound which appears in the proof of the irreducibility of boundary representations. Moreover this uniform bound can be used to prove that the Harish-Chandra's Schwartz space associated with some discrete groups of isometries of δ -hyperbolic spaces carries a natural structure of a convolution algebra. Then in the context of CAT(-1) spaces we show how our elementary techniques enable us to apply an equidistribution theorem of T. Roblin to obtain informations about the decay of matrix coefficient of boundary representations associated with continuous functions.

1. INTRODUCTION

On the unit disc \mathbb{D} , the Poisson transform of a function f in $L^1(\partial\mathbb{D}, \frac{d\theta}{2\pi})$ is defined by the formula

$$Pf(z) = \int_{\partial\mathbb{D}} f(\theta) P(z, \theta) \frac{d\theta}{2\pi},$$

where $P(z, \theta)$ is the standard Poisson kernel and $\frac{d\theta}{2\pi}$ is the normalized Lebesgue measure on the unit circle. It is well known that the harmonic function Pf converges to f for almost all $\theta \in \partial\mathbb{D}$ as $z \rightarrow e^{i\theta}$ and z stays in the nontangential cone which one can define by the inequality $|\arg(z) - \theta| < C(1 - |z|)$ where C is a positive constant.

However, the situation is different when we consider *the (normalized) square root of the Poisson kernel*. Let f in $L^1(\partial\mathbb{D}, \frac{d\theta}{2\pi})$ and define the function $\mathcal{P}_0 f$ by the formula

$$\mathcal{P}_0 f(z) = \int_{\partial\mathbb{D}} f(\theta) \frac{P^{\frac{1}{2}}(z, \theta)}{\int_{\partial X} P^{\frac{1}{2}}(z, \theta) \frac{d\theta}{2\pi}} \frac{d\theta}{2\pi}.$$

P. Sjögren proved in a short note [28] that the function $\mathcal{P}_0 f$ converges to f for almost all $\theta \in \partial\mathbb{D}$ as $z \rightarrow e^{i\theta}$ and z stays in the weak nontangential domain which one can define by the inequality

$$|\arg(z) - \theta| < C(1 - |z|) \left(\log \frac{1}{1 - |z|} \right)$$

for some positive constant $C > 0$.

These convergence results, also called *theorem à la Fatou*, have been studied in [27], [25] and generalized to the context of symmetric spaces in [20], [26] and in [28]. Generalizations of theorems *à la Fatou* have already been done in the context of CAT(-1) spaces or negative curvature as we can see in [23] or in [1]. We generalize this *theorem à la Fatou for the normalized*

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square root of the Poisson kernel to the context of Patterson-Sullivan density associated with quasi-convex cocompact discrete group of isometries of δ -hyperbolic spaces. More precisely, this work is based on the framework of Patterson-Sullivan measures developed by M. Coornaert in [9] for general δ -hyperbolic spaces. Besides, the Poisson kernel being the conformal factor of an isometry acting on the boundary, we make connections with boundary representations and this theorem à la Fatou. We obtain information about decay of matrix coefficients associated with boundary representations and we will also deal with analogs of *Harish-Chandra's function*, *Harish-Chandra's weak inequality* and *Harish-Chandra's Schwartz algebras* in the context of discrete groups of isometries of δ -hyperbolic spaces.

1.1. Notation and statement of the results. Let Γ be a nonelementary discrete group of isometries of a metric space (X, d) which is δ -hyperbolic. We denote by ∂X its Gromov boundary and let \overline{X} be the topological space $X \cup \partial X$ endowed with its usual topology that makes \overline{X} compact.

It is well known (see [16] and [3, Chapter III.H, Proposition 3.21], and see Section 2 in this paper) that the topology on ∂X is metrizable: once a base point x in X has been fixed, there exists a parametrized family $(d_x^\epsilon)_{\epsilon>0}$ of metrics each of which are called a *visual metric* of visual parameter ϵ .

Recall the critical exponent $\alpha(\Gamma)$ of Γ :

$$\alpha(\Gamma) := \inf \left\{ s \in \mathbb{R}_+^* \mid \sum_{\gamma \in \Gamma} e^{-sd(\gamma x, x)} < \infty \right\}.$$

Notice that the definition of $\alpha(\Gamma)$ does not depend on x . We assume from here on now that $\alpha(\Gamma) < \infty$.

The limit set of Γ denoted by Λ_Γ is the set of all accumulation points in ∂X of an orbit. Namely $\Lambda_\Gamma := \overline{\Gamma x} \cap \partial X$, with the closure in \overline{X} . Notice that the limit set does not depend on the choice of $x \in X$. Following the notations in [10], define the geodesic hull $GH(\Lambda_\Gamma)$ as the union of all geodesics in X with both endpoints in Λ_Γ . The convex hull of Λ_Γ denoted by $CH(\Lambda_\Gamma)$, is the smallest subset of X containing $GH(\Lambda_\Gamma)$ with the property that every geodesic segment between any pair of points x and y in $CH(\Lambda_\Gamma)$ also lies in $CH(\Lambda_\Gamma)$. We say that Γ is *quasi-convex cocompact* (respectively *convex cocompact*) if it acts cocompactly on $GH(\Lambda_\Gamma)$ (respectively on $CH(\Lambda_\Gamma)$).

The foundations of Patterson-Sullivan measures theory are in the important papers [21] and [29]. See [4],[5] and [23] for more general results in the context of CAT(-1) spaces. These measures are also called *conformal densities*. However, in this paper we will consider a more general measure class, called *quasi-conformal densities*.

We denote by $M(Z)$ the Banach space of Radon measures on a locally compact space Z , which is identified as the dual space of compactly supported functions denoted by $C_c(Z)^*$, endowed with the norm $\|\mu\| = \sup\{|\int_Z f d\mu|, \|f\|_\infty \leq 1, f \in C_c(Z)\}$ where $\|f\|_\infty = \sup_{z \in Z} |f(z)|$. Recall that $\gamma_*\mu$ means $\gamma_*\mu(B) = \mu(\gamma^{-1}B)$ where γ is in Γ and B is a borel subset of Z .

Definition 1.1. We say that μ is a Γ -invariant quasi-conformal density of dimension $\alpha \geq 0$, if μ is a map which satisfies the following conditions:

- (1) μ is a map from $x \in X \mapsto \mu_x \in M(\overline{X})$, i.e. μ_x is a positive finite measure (density).
- (2) There exists $C_q > 0$ such that for all x and y in X , μ_x and μ_y are equivalent and they satisfy almost everywhere:

$$C_q^{-1} e^{\alpha \beta_v(x,y)} \leq \frac{d\mu_y}{d\mu_x}(v) \leq C_q e^{\alpha \beta_v(x,y)}$$

(quasi-conformal of dimension α).

- (3) For all $\gamma \in \Gamma$ and for all $x \in X$ we have $\gamma_*\mu_x = \mu_{\gamma x}$ (invariant),

where $\beta_v(x, y)$ denotes a Busemann function associated with x to y relative to v (see Section 2).

Moreover if

$$(x, y, v) \in X \times X \times \partial X \mapsto \frac{d\mu_y}{d\mu_x}(v)$$

is continuous, we say that μ is a Γ -invariant continuous quasi-conformal density of dimension $\alpha \geq 0$.

If in (2) we have $C_q = 1$ and that the equality holds everywhere, we say that μ is Γ -invariant conformal density of dimension $\alpha \geq 0$.

In [9, Théorème 8.3], M. Coornaert proves the existence of such Γ -invariant quasi-conformal density of dimension $\alpha(\Gamma)$ when X is a proper geodesic δ -hyperbolic space and when the group Γ is quasi-convex cocompact (as it has been mentioned, after a slight modification, by C. Connell and R. Muchnik [10, Proposition 1.23 and Corollary 1.24]).

If X is a CAT(-1) space and if Γ is a discrete group of isometries of X , then there exists a Γ -invariant conformal density of dimension $\alpha(\Gamma)$ whose support is Λ_Γ . A proof of this theorem can be found in [21] and [29] for the case of hyperbolic spaces and see [6] and [4] for the case of CAT(-1) spaces.

The Poisson kernel. Given μ a Γ -invariant quasi-conformal density of dimension α we define the Poisson kernel in the context of δ -hyperbolic spaces associated with the measure class μ as:

$$(1.1) \quad P : (x, y, v) \in X \times X \times \partial X \mapsto P(x, y, v) := \left(\frac{d\mu_y}{d\mu_x}(v) \right)^{1/\alpha} \in \mathbb{R}^+.$$

Remark 1.2. In the case of CAT(-1) spaces, we have $P(x, y, v) = e^{\beta_v(x, y)}$. Thus in the case of hyperbolic spaces, we find the standard Poisson kernel. Indeed this definition of the Poisson kernel is suggested by the definition of the Poisson kernel as the conformal factor in [4, (2.7.1) p. 92] or as in [21].

We fix an origin $x \in X$ and imitating the notations of Sjögren ([27]) we set for $\lambda \in \mathbb{R}$ and $f \in L^1(\partial X, \mu_x)$

$$P_{\lambda, \mu_x} f(y) := \int_{\partial X} P(x, y, v)^{\alpha(\lambda+1/2)} f(v) d\mu_x(v).$$

Definition 1.3. Define for $\lambda \in \mathbb{R}$ the normalized Poisson kernel as

$$(\mathcal{P}_{\lambda, \mu_x} f) : y \in X \mapsto \frac{P_{\lambda, \mu_x} f(y)}{P_{\lambda, \mu_x} \mathbf{1}_{\partial X}(y)} \in \mathbb{C}.$$

where $\mathbf{1}_{\partial X}$ is the characteristic function of ∂X and y is in X .

In this article, for $\lambda = 0$ we will write $\mathcal{P}_0 f$ rather than $\mathcal{P}_{0, \mu_x} f$ and we will define and study the weak radial convergence called *weak nontangential convergence* of $\mathcal{P}_0 f$.

Upper Gromov product bounded by above and weak nontangential convergence.

To define *weak nontangential convergence* on \overline{X} we use the notion of *upper Gromov product bounded by above*, notion which has already been emphasized in [10]. Following [10, Definition 1.18], we say that a δ -hyperbolic space X is *upper Gromov product bounded by above* if there exists a point $x \in X$ and a positive constant c such that for all $y \in X$:

$$(1.2) \quad \sup_{v \in \partial X} (y, v)_x \geq d(x, y) - c,$$

where $(y, z)_x$ denotes the Gromov product of $y, z \in \overline{X}$ with respect to $x \in X$.

Notice that if the above inequality holds for some point x , the same inequality holds for all x in X (with another constant c).

This property on X enables us to choose a point

$$(1.3) \quad w_x^y \in \partial X,$$

such that

$$(1.4) \quad (y, w_x^y)_x \geq d(x, y) - c.$$

The point w_x^y plays the role of the unique ending point of the oriented geodesic passing through x and y in the context of connected negatively curved manifolds.

Definition 1.4. Let μ be a Γ -invariant quasi-conformal density of dimension α . Fix an origin $x \in X$ and let $v \in \Lambda_\Gamma$. We define a weak nontangential approach domain of v as

$$\Omega_C(v) := \{y \in GH(\Lambda_\Gamma) \mid d_x^\epsilon(w_x^y, v) \leq Cd(x, y)^{\frac{\epsilon}{\alpha}} e^{-\epsilon d(x, y)}\}.$$

We say that a function $F : X \rightarrow \mathbb{C}$ has a weak nontangential limit L at $v \in \partial X$ if for all $C > 0$ we have $F(y) \rightarrow L$ as $y \rightarrow v$ with respect to the topology on \overline{X} and if $y \in \Omega_C(v)$.

We obtain a theorem à la Fatou for the square root of the Poisson kernel in the context of discrete groups of isometries of δ -hyperbolic spaces:

Theorem 1.5. Let X be a proper geodesic δ -hyperbolic space which is upper Gromov product bounded by above and assume that the Gromov boundary ∂X is endowed with a visual metric $(d_x^\epsilon)_{x \in X}$ of visual parameter $0 < \epsilon \leq 1$. Let Γ be a discrete quasi-convex cocompact group of isometries of X and let μ be a Γ -invariant continuous quasi-conformal density of dimension $\alpha(\Gamma)$.

For each x in X and for all $f \in L^1(\partial X, \mu_x)$ the function $\mathcal{P}_0 f$ has weak non tangential limit f almost everywhere with respect to μ_x .

We obtain immediately the following corollary:

Corollary 1.6. Let X be a proper $CAT(-1)$ space and let Γ be a convex cocompact discrete group of isometries of X . Let $\mu = (\mu_x)_{x \in X}$ be a Γ -invariant conformal density of dimension $\alpha(\Gamma)$.

For each x in X and for all $f \in L^1(\partial X, \mu_x)$ we have $\mathcal{P}_0 f$ has weak non tangential limit f almost everywhere with respect to μ_x .

The connection with boundary representations will be almost immediate.

Boundary representations. A Γ -invariant quasi-conformal density μ of dimension α gives rise to unitary representations $(\pi_x)_{x \in X}$ defined for all $x \in X$ as:

$$\begin{aligned} \pi_x : \Gamma &\rightarrow \mathcal{U}(L^2(\partial X, \mu_x)) \\ (1.5) \quad (\pi_x(\gamma)\xi)(v) &= P(x, \gamma x, v)^{\alpha/2} \xi(\gamma^{-1}v), \end{aligned}$$

where $\xi \in L^2(\partial X, \mu_x)$ and $v \in \partial X$.

Remark 1.7. In $CAT(-1)$ spaces these representations $(\pi_x)_{x \in X}$ are unitarily equivalent. More generally, the boundary representations depend only on the measure class from which they are built.

We will focus our attention on some $x \in X$

$$\pi_x : \Gamma \rightarrow \mathcal{U}(L^2(\partial X, \mu_x))$$

where x is an origin in X .

Harish-Chandra's function and growth. One has to define a function on Γ which is going to play a central role in this paper. The matrix coefficient

$$(1.6) \quad \phi_x : \Gamma \rightarrow \langle \pi_x(\gamma)\mathbf{1}_{\partial X}, \mathbf{1}_{\partial X} \rangle \in \mathbb{R}^+,$$

where $\mathbf{1}_{\partial X}$ is the characteristic function of ∂X , is called the *Harish-Chandra function*. Observe that the Harish-Chandra functions satisfies for all γ in Γ

$$(1.7) \quad \phi_x(\gamma) = \phi_x(\gamma^{-1}).$$

We call it the Harish-Chandra function because its definition in the context of reductive Lie groups goes back to Harish-Chandra (see [30, Chapter 3, §3.6.1]).

Following [13, Definition 6.1.17], we say that a function $f : \Gamma \rightarrow \mathbb{C}$ satisfies the *weak inequality of Harish-Chandra of degree t* if for some x in X , there exist $C > 0$ and $t \geq 0$ such that

$$(1.8) \quad |f(\gamma)| \leq C(1 + |\gamma|_x)^t \phi_x(\gamma),$$

where $|\cdot|_x$ is a length function on Γ defined as $|\gamma|_x := d(x, \gamma x)$.

Indeed, it is not hard to see that if the weak inequality holds for some x , then it holds for all x in X with a constant C depending on x but with the same t (see Section 2).

As a corollary of Theorem 1.5, we have:

Corollary 1.8. *Let Γ be a convex cocompact discrete group of isometries of a CAT(-1) space X and let μ be a Γ -invariant conformal density of dimension $\alpha(\Gamma)$. For each x in X , consider the boundary representations $\pi_x : \Gamma \rightarrow \mathcal{U}(L^2(\partial X, \mu_x))$.*

For each x in X and for all ξ in $L^2(\partial X, \mu_x)$ such that ξ is not in $L^\infty(\partial X)$, the matrix coefficient $\langle \pi_x(\cdot) \mathbf{1}_{\partial X}, \xi \rangle$ does not satisfy the weak inequality of Harish-Chandra of degree 0.

Fix a point x in X and denote by φ_x the function

$$(1.9) \quad \varphi_x : y \in X \mapsto P_{0, \mu_x} \mathbf{1}_{\partial X}(y) = \int_{\partial X} P(x, y, v)^{\frac{\alpha}{2}} d\mu_x(v).$$

Observe that the Harish-Chandra function ϕ_x is the restriction of φ_x to the orbit Γx .

Let \mathcal{Y} be a subset of X . We say that φ_x satisfies the *Harish-Chandra estimates on \mathcal{Y}* if there exist two polynomials Q_1 and Q_2 of degree one with positive coefficients such that for all $y \in \mathcal{Y}$ we have

$$(1.10) \quad Q_1(d(x, y))e^{-\frac{\alpha}{2}d(x, y)} \leq \varphi_x(y) \leq Q_2(d(x, y))e^{-\frac{\alpha}{2}d(x, y)}.$$

Define the class

$$(1.11) \quad \mathcal{C} \text{ of discrete groups as the class made of:}$$

- convex cocompact groups of isometries of a CAT(-1) space,
- nonuniform lattices in noncompact semisimple Lie group of rank one acting by isometries on their rank one symmetric spaces of noncompact type where d is a left invariant Riemannian metric,
- hyperbolic groups Γ acting by isometries on (Γ, d_S) , where d_S is the word metric associated with a finite symmetric generating set S .

Remark 1.9. *The common point of these groups is that there exists a Γ -invariant quasi-conformal density of dimension $\alpha(\Gamma)$ denoted by μ such that the function φ_x associated with μ_x satisfies the Harish-Chandra estimates on $GH(\Lambda_\Gamma) \setminus B_X(x, R)$ where $B_X(x, R)$ denotes the ball of X centered at x of radius R .*

We say that a function f defined on a group Γ endowed with a length function l has polynomial growth if there exists a polynomial Q such that for all integers n

$$\sum_{l(\gamma) \leq n} f(\gamma) \leq Q(n).$$

Notice that if a function f on $\Gamma \in \mathcal{C}$ satisfies the weak inequality of Harish-Chandra of degree t with respect to some Harish-Chandra function ϕ_x (associated with the Γ invariant quasi-conformal density of dimension $\alpha(\Gamma)$) then the function $|f|^2$ has polynomial growth with respect to the length function $|\cdot|_x$. In particular the square of Harish-Chandra's function has polynomial growth.

Although Corollary 1.8 asserts that some coefficients cannot decay too rapidly to 0, we are able to show that these coefficients have only polynomial growth. Let $\Gamma_n(x)$ be the ball of Γ of radius n centered at x defined as

$$\{\gamma \in \Gamma \text{ such that } |\gamma|_x \leq n\}.$$

Proposition 1.10. *Let Γ be in \mathcal{C} and consider their boundary representations associated with the Γ -invariant quasi-conformal density of dimension $\alpha(\Gamma)$. Then for each x in X there exists a polynomial Q (of degree 3) such that for all unit vectors ξ in $L^2(\partial X, \mu_x)$ we have for all integers n*

$$\sum_{\gamma \in \Gamma_n(x)} |\langle \pi_x(\gamma) \mathbf{1}_{\partial X}, \xi \rangle|^2 \leq Q(n).$$

Besides, a direct application of an equidistribution theorem of T. Roblin gives another information about the sum of matrix coefficients associated with continuous functions (for a less general class of groups). We denote by $\|\cdot\|_2$ the L^2 norm of $L^2(\partial X, \mu_x)$. Define the annulus $C_{n,\rho}(x)$ for $\rho > 0$ and $n \geq \rho$ as

$$(1.12) \quad \{\gamma \in \Gamma \text{ such that } n - \rho \leq |\gamma|_x \leq n + \rho\}.$$

We refer to [23, Chapitre 1 Préliminaires, 1C. Flot géodésique] and [8, Preliminaries 2.2] for details on the Bowen-Margulis-Sullivan measure and on the mixing property of the geodesic flow. We obtain:

Proposition 1.11. *Let Γ be a discrete group of isometries belonging to one of the following classes:*

- *convex cocompact discrete groups of isometries of a CAT(-1) space such that the geodesic flow is mixing with respect to the Bowen-Margulis-Sullivan measure or*
- *nonuniform lattices in noncompact rank one semisimple Lie groups acting by isometries on their symmetric spaces of noncompact type.*

For each $x \in X$ there exist $\rho > 0$ and a polynomial Q (of degree 3) such that for all continuous functions f and g in $C(\partial X)$ we have

$$\limsup_{n \rightarrow +\infty} \frac{1}{Q(n)} \sum_{\gamma \in C_{n,\rho}(x)} |\langle \pi_x(\gamma) f, g \rangle|^2 \leq \|f\|_2^2 \|g\|_2^2.$$

Remark 1.12. *Proposition 1.10 and Proposition 1.11 are intimately related to the so-called property RD (rapid decay) introduced by U. Haagerup in [17] and studied as such by P. Jolissaint in [18]. The inequality of Proposition 1.10 can be seen as a particular case of the inequality coming from property RD, with one of the two functions being trivial. In Proposition 1.11 if instead the lim sup we had the sup, then we would have obtained property RD.*

Harish-Chandra's Schwartz algebra of discrete groups. Let Γ be a discrete group endowed with a length function denoted by $|\cdot|$ and let $\phi : \Gamma \rightarrow \mathbb{R}^+$ be a positive function on Γ . Assume that ϕ satisfies:

- (1) $\phi(e) = 1$,
- (2) $\phi(\gamma^{-1}) = \phi(\gamma)$, $\forall \gamma \in \Gamma$.
- (3) There exists $t > 0$ such that

$$\sum_{\gamma \in \Gamma} \frac{\phi^2(\gamma)}{(1 + |\gamma|)^t} < \infty.$$

We denote by $C_c(\Gamma)$ the space of finitely supported functions.

Definition 1.13. *Let $t > 0$ and define Harish-Chandra's Schwartz space $\mathcal{S}_t(\Gamma)$ associated with $(\Gamma, |\cdot|, \phi)$ as the Banach space completion of $C_c(\Gamma)$ with respect to the norm*

$$\|f\|_{\mathcal{S}_t(\Gamma)} = \sup_{\gamma \in \Gamma} \frac{|f(\gamma)|}{\phi(\gamma)} (1 + |\gamma|)^t.$$

Observe that if a function $f : \Gamma \rightarrow \mathbb{C}$ satisfies the weak inequality of degree $t \geq 0$ (with respect to ϕ), then $f \in \mathcal{S}_t(\Gamma)$.

Notice that assumption (3) guarantees that $\mathcal{S}_t(\Gamma) \subset \ell^2(\Gamma)$.

We can describe $\mathcal{S}_t(\Gamma)$ as

$$\mathcal{S}_t(\Gamma) = \left\{ \sum_{\gamma \in \Gamma} c_\gamma \gamma \text{ such that } c_\gamma = O\left(\frac{\phi(\gamma)}{(1 + |\gamma|)^t}\right) \right\}.$$

It turns out that Harish-Chandra's Schwartz space carries a natural structure of a convolution algebra and can be represented on $\ell^2(\Gamma)$. We denote by $C_r^*(\Gamma)$ the reduced C^* -algebra of Γ that is the C^* -algebra associated with the left regular representation of Γ and by $W^*(\Gamma)$ the von Neumann algebra of Γ that is the von Neumann algebra associated with the left regular representation of Γ .

Proposition 1.14. *Let Γ be in \mathcal{C} and consider a Γ -invariant quasi-conformal density $(\mu_x)_{x \in X}$ of dimension $\alpha(\Gamma)$. Then for each $x \in X$, there exists $t > 0$ such that Harish-Chandra's Schwartz space $\mathcal{S}_t(\Gamma)$ associated with $(\Gamma, |\cdot|_x, \phi_x)$ is a $*$ -Banach convolution algebra which is dense in $C_r^*(\Gamma)$ and in $W^*(\Gamma)$.*

1.2. Structure of the paper. In Section 2 we remind the reader of some standard facts about δ -hyperbolic spaces, Ahlfors regularity of metric measure space, Harish-Chandra's function and maximal inequality of type (1, 1).

In Section 3 we prove our main result: Theorem 1.5.

In Section 4 we deduce immediately Corollary 1.8 from Theorem 1.5. We recall a uniform boundedness satisfied by a certain sequence of functions which has already been used in the proof of the irreducibility of boundary representations (in [6], [15] and [8]). We use this uniform boundedness to prove Proposition 1.10. We recall a theorem of equidistribution of T. Roblin and combining this theorem with techniques involved in the proof of Theorem 1.5 we prove Proposition 1.11.

In Section 5 we use the uniform boundedness of the previous section to prove Proposition 1.14. It turns out that the inequality of Proposition 4.3 is fundamental to prove that the Harish-Chandra's Schwartz space of discrete groups carries a natural structure of convolution algebra. This inequality plays analogous roles of the conditions (HC3c) and (HC3d) in [19, Chapitre 4, p. 82] satisfied by spherical functions in the context of semisimple Lie groups.

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2. PRELIMINARIES

2.1. δ -Hyperbolic spaces. We follow [3, Chapter III.H, p. 431] and [14], after [16].

Recall that the Gromov product of two points $y, z \in X$ relative to $x \in X$ is

$$(y, z)_x = \frac{1}{2}(d(x, y) + d(x, z) - d(y, z)).$$

We say that (X, d) is δ -hyperbolic if for x, y, z and u in X we have

$$(y, z)_x \geq \min\{(y, u)_x, (u, z)_x\} - \delta.$$

A sequence $(a_n)_{n \in \mathbb{N}}$ in X converges at infinity if $(a_i, a_j)_x \rightarrow +\infty$ as i, j goes to $+\infty$. We say that two sequences $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ are equivalent if $(a_i, b_j)_x \rightarrow +\infty$ as i, j goes to ∞ . An equivalence class of $(a_n)_{n \in \mathbb{N}}$ is denoted by $\lim a_n$ and we denote by ∂X the set of equivalence classes. These definitions are independent of the choice of the basepoint x .

We extend the definition of the Gromov product to \overline{X} by

$$(2.1) \quad (v, w)_x := \sup_{i, j} \lim (a_i, b_j)_x$$

where the sup is taken over all sequences $(a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}}$ such that $v = \lim_i a_i$ and $w = \lim_j b_j$.

To sum up the property of the Gromov product in general δ -hyperbolic spaces we have:

Proposition 2.1. [3, 3.17 Remarks, p. 433].

Let X be a δ -hyperbolic space and fix x in X .

- (1) The extended Gromov product $(\cdot, \cdot)_x$ is continuous on $X \times X$, but not necessarily on $\overline{X} \times \overline{X}$.
- (2) In the definition of $(a, b)_x$, if we have a in X (or b in X), then we may always take the respective sequence to be the constant value $a_i = a$ (or $b_j = b$).
- (3) For all v, w in \overline{X} there exist sequences (a_n) and (b_n) such that $v = \lim a_n$ and $w = \lim b_n$ and $(v, w)_x = \lim_{n \rightarrow +\infty} (a_n, b_n)_x$.
- (4) For all u, v and w in \overline{X} by taking limits we still have

$$(v, w)_x \geq \min \{(v, u)_x, (u, w)_x\} - 2\delta.$$

- (5) For all v, w in ∂X and all sequences (a_i) and (b_j) in X with $v = \lim a_i$ and $w = \lim b_j$, we have:

$$(v, w)_x - 2\delta \leq \liminf_{i,j} (a_i, b_j)_x \leq (v, w)_x.$$

We refer to [14, 8.- Remarque, Chapitre 7, p. 122] for a proof of the statement (5) which is also an exercise, but it will be very useful in this note.

Indeed the Gromov boundary of a proper geodesic space has also a geometrical definition ([3, Chapter III.H, p. 431]) and if $v \in \partial X$ we can pick a geodesic ray c , namely a map $r : \mathbb{R}_+ \rightarrow X$ which is an isometry, such that $r(+\infty) = v$ to define the *Busemann function associated to r* as

$$(2.2) \quad b_r(x) = \lim_{t \rightarrow +\infty} d(x, r(t)) - t,$$

which is well defined due to the triangle inequality.

We define the *horospherical distance relative to r* as:

$$(2.3) \quad B_r(x, y) = b_r(x) - b_r(y).$$

where c is a geodesic ray such that $v = r(+\infty)$. Following the vocabulary of [10, Definition 1.21] we define the *Busemann function* as

$$(2.4) \quad \beta_v(x, y) = 2(v, y)_x - d(x, y).$$

It follows from Proposition 2.1 (5) that for all r such that $r(+\infty) = v$

$$\beta_v(x, y) - 4\delta \leq B_r(x, y) \leq \beta_v(x, y).$$

We have the following properties concerning the Busemann functions:

Proposition 2.2. We have for $x, y, z \in X$ and $v \in \partial X$:

- (1) $0 \leq \beta_v(x, y) + \beta_v(y, x) \leq 4\delta$.
- (2) $-d(x, y) \leq \beta_v(x, y) \leq d(x, y) + 2\delta$.
- (3) $\beta_v(x, z) \leq \beta_v(x, y) + \beta_v(y, z) \leq \beta_v(x, z) + 4\delta$.

Recall the notion of visual metric on the boundary: fix a basepoint x and let $\epsilon \leq \frac{\log 2}{4\delta}$. Then there exists a metric d_x^ϵ on the boundary such that for all $v, w \in \partial X$

$$(2.5) \quad \frac{1}{c_m} e^{-\epsilon(v, w)_x} \leq d_x^\epsilon(v, w) \leq c_m e^{-\epsilon(v, w)_x}$$

for some positive constant $c_m > 0$, see [14, Chapitre 7, §3] for more details.

A particular class of δ -hyperbolic spaces is the class of CAT(-1) spaces. A CAT(-1) space is a proper metric geodesic space such that every geodesic triangle is thinner than its comparison triangle in the hyperbolic plane. For example, every complete simply connected Riemannian manifolds of curvature $\kappa < 0$ is a CAT(-1) space. In the context of CAT(-1) spaces, the formula

$$(2.6) \quad d_x(v, w) = e^{-(v, w)_x}$$

defines a distance on ∂X (we set $d_x(v, v) = 0$). This is due to M. Bourdon, we refer to [4, 2.5.1 Théorème] for more details.

Once a visual parameter ϵ has been chosen and a basepoint x in X has been fixed, a ball of radius $r > 0$ centered at $v \in \partial X$ with respect to d_x^ϵ is denoted by $B(v, r)$. A ball of radius r centered at $x \in X$ is denoted by $B_X(x, r)$.

As we said in the introduction, if X is *upper gromov bounded by above*, we can choose $w_x^y \in \partial X$ such that $(v, w_x^y)_x \geq d(x, y) - c$. In particular CAT(-1) spaces are upper Gromov bounded by above and also the hyperbolic groups (viewed as metric spaces) endowed with a left invariant word metric associated with some finite symmetric set of generators (see [15, Lemma 4.1])

Lemma 2.3. *Assume that X is upper Gromov bounded by above. There exists $c > 0$ such that for all $\gamma \in \Gamma$, we have*

- (1) $(v, y)_x \leq (v, w_x^y) + c + 2\delta$.
- (2) $\min\{(v, w_x^y), d(x, y)\} - c - 2\delta \leq (v, y)_x$.

Proof. We prove (1). We have by Proposition 2.1 (4)

$$\begin{aligned} (v, w_x^y)_x &\geq \min\{(v, y)_x, (y, w_x^y)_x\} - 2\delta \\ &\geq (v, y)_x - c - 2\delta. \end{aligned}$$

□

The radial limit set Λ_Γ^{rad} is a subset of Λ_Γ such that a point $v \in \Lambda_\Gamma^{rad}$ if and only if there exists a positive constant $C > 0$ and a sequence $(\gamma_n)_{n \in \mathbb{N}}$ with $(\gamma_n y)_{n \in \mathbb{N}}$ converging to v such that

$$(2.7) \quad d(\gamma_n y, x) - (\gamma_n y, v)_x \leq C,$$

Notice that the definition does not depend on x or y . We have:

Lemma 2.4. *Assume that X is upper Gromov product bounded by above. If $v \in \Lambda_\Gamma^{rad}$ there exist $C > 0$ and a sequence (γ_n) in Γ such that $(\gamma_n x) \in \Omega_C(v)$.*

Proof. Let x be in $GH(\Lambda_\Gamma)$ and let v be in Λ_Γ^{rad} . There exist $C' > 0$ and a sequence $(\gamma_n) \in \Gamma$ such that

$$(2.8) \quad d(\gamma_n x, x) - (\gamma_n x, v)_x \leq C'.$$

We can assume that $d(x, \gamma_n x) \geq 1$ for all n . For all n consider the point $w_x^{\gamma_n x}$ in ∂X and apply Lemma 2.3 (1) to obtain

$$(\gamma_n x, v)_x \leq (w_x^{\gamma_n x}, v)_x + c + 2\delta.$$

Combining the above inequality with Inequality (2.8) and with Inequality (2.5) we obtain:

$$\begin{aligned} d_x^\epsilon(v, w_x^{\gamma_n x}) &\leq c_m e^{\epsilon(C' + c + 2\delta)} e^{-\epsilon d(x, \gamma_n x)} \\ &\leq c_m e^{\epsilon(C' + c + 2\delta)} d(x, \gamma_n x)^{\epsilon/\alpha} e^{-\epsilon d(x, \gamma_n x)}, \end{aligned}$$

and by Definition 1.4, $(\gamma_n x) \in \Omega_C(v)$ with the constant $C = c_m e^{\epsilon(C' + c + 2\delta)}$. This completes the proof. □

It is well known that for quasi-convex cocompact groups we have $\Lambda_\Gamma = \Lambda_\Gamma^{rad}$, see for example [23, Chapitre 1, Préliminaires 1F, Proposition 1.10]. Notice also in CAT(-1) spaces that $GH(\Lambda_\Gamma) = CH(\Lambda_\Gamma)$.

As we have mentioned in the introduction, if the weak inequality of Harish-Chandra holds for some x then it holds for all x in X . Indeed observe that for all x and y ,

$$1 + |\gamma|_x \leq (1 + 2d(x, y))(1 + |\gamma|_y)$$

and observe also that Proposition 2.2 implies

$$\forall \gamma \in \Gamma, \phi_x(\gamma) \leq C \phi_y(\gamma),$$

for some positive constant $C > 0$ which does not depend on γ .

2.2. Standard facts about measure theory.

2.2.1. Ahlfors regularity and Harish-Chandra function. Let (Z, d, m) be a compact metric measure space with a metric d and a measure m . We denote by $\text{Diam}(Z)$ the diameter of Z . We say that the metric measure space Z is *Ahlfors D -regular* if there exists a positive constant $k > 0$ such that for all z in Z and $0 < r < \text{Diam}(Z)$ we have

$$k^{-1}r^D \leq m(B(z, r)) \leq kr^D.$$

If only the right hand side inequality holds, then we say that (Z, d, m) is *upper Ahlfors D -regular*.

Let (X, d) be a δ -hyperbolic space, let Γ be a quasi-convex cocompact discrete group of isometries of X and let $\mu = (\mu_x)_{x \in X}$ be a Γ -invariant quasi-conformal density of dimension $\alpha(\Gamma)$. Fix an origin x in X and consider the metric measure space $(\partial\Lambda_\Gamma, d_x^\epsilon, \mu_x)$, where d_x^ϵ is a visual metric with visual parameter $\epsilon > 0$. A fundamental property of the conformal densities (which goes back to D. Sullivan in the case of hyperbolic spaces \mathbb{H}^n , see [29]) and which is due to M. Coornaert in general δ -hyperbolic spaces (see [9, Proposition 7.4]) is that the metric measure space $(\Lambda_\Gamma, d_x^\epsilon, \mu_x)$ satisfies Ahlfors regularity conditions. More precisely, in the case of general proper and geodesic δ -hyperbolic spaces, the metric measure space $(\Lambda_\Gamma, d_x^\epsilon, \mu_x)$ is Ahlfors $D(\Gamma)$ -regular where $D(\Gamma) = \alpha(\Gamma)/\epsilon$. Using the Ahlfors $D(\Gamma)$ -regularity of $(\Lambda_\Gamma, d_x^\epsilon, \mu_x)$, we can obtain the Harish-Chandra estimates as follows:

Proposition 2.5. *Assume that X is upper Gromov bounded by above. Let $\mu = (\mu_x)_{x \in X}$ be a Γ -invariant conformal density of dimension $\alpha(\Gamma)$, where Γ is a quasi-convex cocompact discrete group of isometries of X . Then for each x in X there exists $R > 0$ such that the function φ_x satisfies the Harish-Chandra estimates on $\Gamma x \setminus B_X(x, R)$.*

Moreover, for each x in X there exists $R > 0$ such that φ_x satisfies the Harish-Chandra estimates on $\text{GH}(\Lambda_\Gamma) \setminus B_X(x, R)$.

The arguments coming from [6, Proposition 3.3] combined with the assumption of *upper Gromov bounded by above* give a proof of the Harish-Chandra estimates, as we can see in [8, Proposition 4.4] in CAT(-1) spaces or in [15, Lemma 5.1] in the case of hyperbolic groups.

The Harish-Chandra estimates are on $\text{GH}(\Lambda_\Gamma) \setminus B_X(x, R)$ and not on $\text{GH}(\Lambda_\Gamma)$ because we want to obtain polynomials with positive coefficients.

2.2.2. Maximal inequality of type (1,1). Let ν be a complex measure and we denote by $\|\nu\|$ its norm. Define the maximal function associated with ν with respect to μ_x as

$$(2.9) \quad \mathcal{M}\nu(v) := \sup_{r>0} \frac{|\nu|(B(v, r))}{\mu_x(B(v, r))},$$

and notice that if $d\nu = f d\mu_x$ with $f \in L^1(\partial X, \mu_x)$ we denote $\mathcal{M}\nu$ by $\mathcal{M}f$ and we have:

$$(2.10) \quad \mathcal{M}f(v) = \sup_{r>0} \frac{1}{\mu_x(B(v, r))} \int_{B(v, r)} |f(w)| d\mu_x(w).$$

2.2.3. Covering, Vitali's lemma and maximal inequality of type (1,1). We follow [22, Chapitre 7] and state the following properties.

Lemma 2.6. *Let (Z, d) be a metric space. Let $A \subset Z$ such that $A \subset \cup_{i \in I} B(z_i, r_i)$ where I is a finite set. Then there exists $I^* \subset I$ such that for all $i \neq j$ in I^* we have $B(z_i, r_i) \cap B(z_j, r_j) = \emptyset$ and $A \subset \cup_{i \in I^*} B(z_i, 3r_i)$.*

And here is the fundamental tool of our main result.

Proposition 2.7. *Maximal inequality of type (1,1).*

Let (Z, d, m) be a metric measure space such that m is a (upper) Ahlfors D -regular measure. Let ν be a complex measure on Z . Then we have

$$m(\{\mathcal{M}\nu > t\}) \leq C 3^D \frac{\|\nu\|}{t}.$$

We refer to [22, Théorème 7.4] for a proof.

3. THEOREM À LA FATOU

3.1. Continuity. We recall a result obtained in [8, Section 7] in the context of CAT(-1) spaces and give the proofs of the results in the slightly more general context of quasi-conformal *continuous* densities.

Let X be proper geodesic δ -hyperbolic space which is upper Gromov product bounded by above and assume that the Gromov boundary ∂X is endowed with visual metrics $(d_x^\epsilon)_{x \in X}$ of visual parameter $\epsilon > 0$. Let Γ be a discrete group of isometries of X and consider μ a Γ -invariant quasi-conformal density of dimension α . Observe that the closure of $GH(\Lambda_\Gamma)$ in \overline{X} satisfies $\overline{GH(\Lambda_\Gamma)} \cap \partial X = \Lambda_\Gamma$. For each $x \in X$ and for all $f \in L^1(\partial X, \mu_x)$ define

$$(3.1) \quad \overline{\mathcal{P}_0}f : y \in \overline{GH(\Lambda_\Gamma)} \mapsto \begin{cases} (\mathcal{P}_0 f)(y) & \text{if } y \in GH(\Lambda_\Gamma) \\ f(y) & \text{if } y \in \Lambda_\Gamma. \end{cases}$$

The goal of this subsection is to prove the following proposition:

Proposition 3.1. *Let μ be a Γ -invariant continuous quasi-conformal density of dimension α and fix an origin x in X . Assume that the function φ_x satisfies the left hand side inequality of the Harish-Chandra estimates. Then for all continuous functions f in $C(\partial X)$ the function $\overline{\mathcal{P}_0}f$ is continuous on $\overline{GH(\Lambda_\Gamma)}$.*

We give a proof of this standard result in the case of the unit disc, in the context of δ -hyperbolic spaces. We follow a standard method based on a sequence of functions called “mollifiers” or called “approximation of the identity”, or also called a Dirac-Weierstraß family.

3.1.1. Dirac-Weierstraß family. We follow [11, Chapter 2, §2.1, p 46] and adapt the definition of a Dirac-Weierstraß family to any finite and positive measure ν .

Definition 3.2. *A Dirac-Weierstraß family $(K(y, \cdot))_{y \in GH(\Lambda_\Gamma)}$ with respect to ν , is a continuous map $K : (y, v) \in GH(\Lambda_\Gamma) \times \Lambda_\Gamma \mapsto K(y, v) \in \mathbb{R}$ satisfying*

- (1) $K(y, v) \geq 0$ for all $v \in \Lambda_\Gamma$ and $y \in GH(\Lambda_\Gamma)$,
- (2) $\int_{\Lambda_\Gamma} K(y, v) d\nu(v) = 1$ for all $y \in GH(\Lambda_\Gamma)$,
- (3) for all $v_0 \in \Lambda_\Gamma$ and for all $r > 0$ we have:

$$\int_{\Lambda_\Gamma \setminus B(v_0, r)} K(y, v) d\nu(v) \rightarrow 0 \text{ as } y \rightarrow v_0.$$

A Dirac-Weierstraß family yields an integral operator \mathcal{K} :

$$\mathcal{K} : f \in L^1(\Lambda_\Gamma, \nu) \mapsto \mathcal{K}f \in C(GH(\Lambda_\Gamma))$$

defined as :

$$\mathcal{K}f : y \in GH(\Lambda_\Gamma) \mapsto \int_{\Lambda_\Gamma} f(v) K(y, v) d\nu(v) \in \mathbb{C}.$$

3.1.2. Continuity. Let f be in $L^1(\Lambda_\Gamma, \nu)$. We define the function $\overline{\mathcal{K}}f$ on $\overline{GH(\Lambda_\Gamma)}$ as the following:

$$(3.2) \quad \overline{\mathcal{K}}f : y \in \overline{GH(\Lambda_\Gamma)} \mapsto \overline{\mathcal{K}}f(y) = \begin{cases} \mathcal{K}f(y) & \text{if } y \in GH(\Lambda_\Gamma) \\ f(y) & \text{if } y \in \Lambda_\Gamma \end{cases}$$

Thus, $\overline{\mathcal{K}}$ is an operator which assigns a function defined on $\overline{GH(\Lambda_\Gamma)}$ to a function defined on Λ_Γ .

Proposition 3.3. *If f is a continuous functions on Λ_Γ , the function $\overline{\mathcal{K}}(f)$ is a continuous function on $\overline{GH(\Lambda_\Gamma)}$.*

Proof. Since K is a continuous function, $\overline{\mathcal{K}}f$ is continuous on $GH(\Lambda_\Gamma)$. Let v_0 be in Λ_Γ .

Let $\varepsilon > 0$ (we use ε for this small quantity because ϵ is reserved for the visual parameter). Since f is continuous, there exists $r > 0$ such that

$$|f(v_0) - f(v)| < \frac{\varepsilon}{2},$$

whenever $v \in B(v_0, r)$. Besides, by (3) in Definition 3.2, there exists a neighborhood V of v_0 such that for all $y \in V$ we have:

$$\int_{\Lambda_\Gamma \setminus B(v_0, r)} K(y, v) d\nu(v) \leq \frac{\varepsilon}{4\|f\|_\infty}.$$

We have for all $y \in V$:

$$\begin{aligned} |\overline{\mathcal{K}}f(v_0) - \overline{\mathcal{K}}f(y)| &= |f(v_0) - \mathcal{K}(y)| = \left| \int_{\Lambda_\Gamma} f(v_0) - f(v) K(y, v) d\nu(v) \right| \\ &\leq \int_{B(v_0, r)} |f(v_0) - f(v)| K(y, v) d\nu(v) + \int_{\Lambda_\Gamma \setminus B(v_0, r)} |f(v_0) - f(v)| K(y, v) d\nu(v) \\ &\leq \varepsilon. \end{aligned}$$

Hence, $\overline{\mathcal{K}}f$ is a continuous function on $\overline{GH(\Lambda_\Gamma)}$. \square

3.1.3. An example of Dirac-Weierstraß family. We shall use the assumption Gromov product upper bounded by above to obtain the following lemma.

Lemma 3.4. *Let v be in Λ_Γ . Then $d_x^\epsilon(v, w_x^y) \rightarrow 0$ as $y \rightarrow v$.*

Proof. Let y_n be a sequence of points of X such that $y_n \rightarrow v$. Lemma 2.3 (1) implies

$$(v, w_x^{y_n})_x \geq (v, y_n)_x - c - \delta.$$

Since $y_n \rightarrow v$, the quantity $(v, y_n)_x$ goes to infinity and thus $d_x^\epsilon(v, w_x^y) \rightarrow 0$ as $y \rightarrow v$. \square

Proposition 3.5. *Fix a base point x in X and consider the finite positive measure μ_x . Assume that there exists a polynomial Q_1 (at least of degree 1) with positive coefficients, such that for all $y \in GH(\Lambda_\Gamma) \setminus B_X(x, R)$ for some $R > 0$ we have*

$$Q_1(d(x, y)) \exp\left(-\frac{\alpha}{2}d(x, y)\right) \leq \varphi_x(y).$$

Then

$$(K(y, \cdot))_{y \in GH(\Lambda_\Gamma)} := \left(\frac{P(x, y, \cdot)^{1/2}}{\varphi_x(y)} \right)_{y \in GH(\Lambda_\Gamma)}$$

is a Dirac-Weierstraß family with respect to μ_x .

Proof. First of all, notice that K is a continuous function by hypothesis on the quasi-conformal density. The statements (1) and (2) in Definition 3.2 are obvious, we shall only prove (3).

Let $B(v_0, r)$ be the ball of radius r at v_0 in ∂X . Let $\varepsilon > 0$ (we use ε for this small quantity because ϵ is reserved for the visual parameter). Since Q_1 is a polynomial at least of degree one, then there exists $R' > 0$ (chosen with $R' \geq R$) such that for all y satisfying $d(x, y) > R'$ we have:

$$\frac{C_{r, \alpha, \delta} \|\mu_x\|}{Q_1(d(x, y))} < \varepsilon$$

where $C_{r, \alpha, \delta} = (2^\alpha e^{\alpha(2\delta+c)} C_q c_m^{\alpha/\epsilon}) / r^\alpha$ is a positive constant and where C_q and c_m come from Definition 1.1 and Inequality (2.5).

Lemma 3.4 yields a neighborhood V of v_0 such that $d_x(v_0, w_x^y) \leq \frac{r}{2}$ for all $y \in V$. We have for all v in $\partial X \setminus B(v_0, r)$:

$$\begin{aligned} d_x^\epsilon(v, w_x^y) &\geq d_x^\epsilon(v, v_0) - d_x^\epsilon(v_0, w_x^y) \\ &\geq r - d_x^\epsilon(v_0, w_x^y) \\ &\geq \frac{r}{2}. \end{aligned}$$

We set $V_{R'} = V \cap X \setminus B_X(x, R')$. Using the assumption *upper Gromov bounded by above* with the above inequality we obtain for all $y \in V_{R'}$:

$$\begin{aligned}
\int_{\partial X \setminus B(v_0, r)} \frac{P(x, y, v)^{\frac{\alpha}{2}}}{\varphi_x(y)} d\mu_x(v) &\leq C_q \int_{\partial X \setminus B(v_0, r)} \frac{e^{\alpha(v, y)_x - \frac{\alpha}{2}d(x, y)}}{\varphi_x(y)} d\mu_x(v) \\
&\stackrel{\text{Lemma 2.3 (1)}}{\leq} C_q c_m^{\alpha/\epsilon} \int_{\partial X \setminus B(v_0, r)} \frac{e^{\alpha(2\delta+c)} e^{-\frac{\alpha}{2}d(x, y)}}{d_x^\epsilon(v, w_x^y)^\alpha \varphi_x(y)} d\mu_x(v) \\
&\leq C_{r, \alpha, \delta} \int_{\partial X \setminus B(v_0, r)} \frac{e^{-\frac{\alpha}{2}d(x, y)}}{Q_1(d(x, y)) e^{-\frac{\alpha}{2}d(x, y)}} d\mu_x(v) \\
&= C_{r, \alpha, \delta} \int_{\partial X \setminus B(v_0, r)} \frac{1}{Q_1(d(x, y))} d\mu_x(v) \\
&\leq \frac{C_{r, \alpha, \delta} \|\mu_x\|}{Q_1(d(x, y))} \\
&\leq \varepsilon.
\end{aligned}$$

Hence

$$\int_{\partial X \setminus B(v_0, r)} \frac{P(x, y, v)^{\frac{\alpha}{2}}}{\varphi_x(y)} d\mu_x(v) \rightarrow 0 \text{ as } y \rightarrow v_0.$$

□

We are ready to prove Proposition 3.1.

Proof of Proposition 3.1. Since Γ is convex cocompact, the Harish-Chandra estimates of φ_x hold on $GH(\Lambda_\Gamma) \setminus B_X(x, R)$ by Proposition 2.5. Hence, Proposition 3.3 combined with Proposition 3.5 complete the proof. □

3.2. Inequality on nontangential maximal function. Let R be a positive real number and define for all $C > 0$ the weak nontangential maximal function associated with $f \in L^1(\partial X, \mu_x)$ as

$$(3.3) \quad \mathcal{N}_C^R f : v \in \partial X \mapsto \sup \{ |\mathcal{P}_0 f(y)| \text{ such that } y \in \Omega_C(v) \setminus B_X(x, R) \} \in \mathbb{R}^+.$$

Here is the fundamental proposition of this paper and the key ingredient is the lower bound of the Harish-Chandra function or in other words the left hand side of Harish-Chandra estimates.

Proposition 3.6. *Assume that the visual parameter satisfies $\epsilon \leq 1$ and assume that φ_x satisfies the Harish-Chandra estimates on $GH(\Lambda_\Gamma) \setminus R$. For all $C > 0$, there exists $R_0 > 0$ and a constant $C_0 > 0$ such that we have*

$$\mathcal{N}_C^{R_0} f \leq C_0 \mathcal{M} f.$$

We follow the proof of [28].

Proof. We will write $|y|$ instead of $d(x, y)$ and define for $v_0 \in \partial X$ and $y \in X$ the ball $B(v_0, \rho(y))$ of ∂X centered at v_0 with the radius

$$\rho(y) := 2C|y|^{\frac{\epsilon}{\alpha}} e^{-\epsilon|y|}.$$

Denote by B_ρ the ball $B(v_0, \rho(y))$ and write $\partial X = B_\rho \cup (\partial X \setminus B_\rho)$. Let a be a positive real number such that $y \in GH(\Lambda_\Gamma) \setminus B_X(x, R)$ implies $\varphi_x(y) \geq Q_1(|y|) \geq a|y|$. Let f be a positive function in $L^1(\partial X, \mu_x)$ and denote by C_q a constant coming from Definition 1.1 (2) of quasi-conformal density. We have for all $y \in GH(\Lambda_\Gamma) \setminus B_X(x, R)$:

$$\begin{aligned}
\frac{1}{C_q} \int_{B_\rho} f(v) \frac{P(x, y, v)^{\frac{\alpha}{2}}}{\varphi_x(y)} d\mu_x(v) &\leq \int_{B_\rho} f(v) \frac{e^{\frac{\alpha}{2}\beta_v(x, y)}}{Q_1(|y|)e^{-\frac{\alpha}{2}|y|}} d\mu_x(v) \\
\text{Proposition 2.2 (2)} &\leq e^{\alpha\delta} \int_{B_\rho} f(v) \frac{e^{\frac{\alpha}{2}|y|}}{Q_1(|y|)e^{-\frac{\alpha}{2}|y|}} d\mu_x(v) \\
&\leq \frac{e^{\alpha\delta}}{Q_1(|y|)e^{-\alpha|y|}} \int_{B_\rho} f(v) d\mu_x(v) \\
&\leq \frac{e^{\alpha\delta}}{a|y|e^{-\alpha|y|}} \int_{B_\rho} f(v) d\mu_x(v) \\
\text{Ahlfors } \alpha(\Gamma)/\epsilon\text{-regularity of } (\Lambda_\Gamma, d_x^\epsilon, \mu_x) &\leq \frac{(2C)^{\alpha/\epsilon} e^{\alpha\delta}}{a \times k} \times \left(\frac{1}{\mu_x(B_\rho)} \int_{B_\rho} f(v) d\mu_x(v) \right) \\
&\leq \left(\frac{(2C)^{\alpha/\epsilon} e^{\alpha\delta}}{a \times k} \right) \mathcal{M}f(v_0),
\end{aligned}$$

Hence for all $y \in GH(\Lambda_\Gamma) \setminus B_X(x, R)$:

$$(3.4) \quad \frac{1}{C_q} \int_{B_\rho} f(v) \frac{P(x, y, v)^{\frac{\alpha}{2}}}{\varphi_x(y)} d\mu_x(v) \leq \left(\frac{(2C)^{\alpha/\epsilon} e^{\alpha\delta}}{ak} \right) \mathcal{M}f(v_0).$$

If $y \in \Omega_C(v)$ we have $d_x^\epsilon(w_x^y, v_0) \leq C|y|^{\frac{\epsilon}{\alpha}} \exp(-\epsilon|y|) = \frac{1}{2}\rho(y)$. Thus, for v satisfying $d_x^\epsilon(v, v_0) \geq \rho(y)$ we have

$$d_x^\epsilon(v, v_0) \leq d_x^\epsilon(v, w_x^y) + \frac{1}{2}d_x^\epsilon(v, v_0),$$

and hence

$$(3.5) \quad d_x^\epsilon(v, w_x^y) \geq \frac{d_x^\epsilon(v, v_0)}{2}.$$

Then, for the second term of the integral on $\partial X \setminus B_\rho$ we have for $y \in GH(\Lambda_\Gamma) \setminus B_X(x, R)$

$$\begin{aligned}
\frac{1}{C_q} \int_{\partial X \setminus B_\rho} f(v) \frac{P(x, y, v)^{\frac{\alpha}{2}}}{\varphi_x(y)} d\mu_x(v) &\leq \int_{d_x^\epsilon(v, v_0) > \rho(y)} f(v) \frac{e^{\alpha(v, y)_x - \frac{\alpha}{2}|y|}}{Q_1(|y|)e^{-\frac{\alpha}{2}|y|}} d\mu_x(v) \\
&= \frac{1}{Q_1(|y|)} \int_{d_x^\epsilon(v, v_0) > \rho(y)} f(v) e^{\alpha(v, y)_x} d\mu_x(v) \\
\text{Lemma 2.3 (1)} &\leq \frac{e^{\alpha(2\delta+c)}}{a|y|} \int_{d_x^\epsilon(v, v_0) > \rho(y)} f(v) e^{\alpha(v, w_x^y)_x} d\mu_x(v) \\
&\leq \frac{e^{\alpha(2\delta+c)} c_m^{\alpha/\epsilon}}{a|y|} \int_{d_x^\epsilon(v, v_0) > \rho(y)} f(v) \frac{1}{d_x^\epsilon(v, w_x^y)^\alpha} d\mu_x(v) \\
\text{Inequality (3.5)} &\leq \frac{2^\alpha e^{\alpha(2\delta+c)} c_m^{\alpha/\epsilon}}{a} \times \frac{1}{|y|} \int_{d_x^\epsilon(v, v_0) > \rho(y)} f(v) \frac{1}{d_x^\epsilon(v, v_0)^\alpha} d\mu_x(v).
\end{aligned}$$

We work now with the term

$$\int_{d_x^\epsilon(v, v_0) > \rho(y)} f(v) \frac{1}{d_x^\epsilon(v, v_0)^\alpha} d\mu_x(v).$$

Define $N(y)$ the integer such that $N(y) = E\left(\frac{\log(D/\rho(y))}{\log 2}\right) + 1$, where D denotes the diameter of $\partial X \setminus B_\rho$ and E denotes the integer part. Take $R' > 0$ such that $|y| \geq R'$ implies $N(y) \leq c'|y|$ for some positive constant $c' > 0$. We may assume that $\|\mu_x\| = 1$ (if not, do the same computation

with $\nu_x = \frac{\mu_x}{\|\mu_x\|}$). We have for $\epsilon \leq 1$:

$$\begin{aligned}
\int_{d_x^\epsilon(v, v_0) > \rho(y)} f(v) \frac{1}{d_x^\epsilon(v, v_0)^\alpha} d\mu_x(v) &= \sum_{k=0}^{N(y)} \int_{2^{k+1}\rho(y) > d_x^\epsilon(v, v_0) > 2^k \rho(y)} f(v) \frac{1}{d_x^\epsilon(v, v_0)^\alpha} d\mu_x(v) \\
&\leq \frac{1}{\rho(y)^\alpha} \sum_{k=0}^{N(y)} \frac{1}{2^{\alpha k}} \int_{2^{k+1}\rho(y) > d_x^\epsilon(v, v_0) > 2^k \rho(y)} f(v) d\mu_x(v) \\
&= 2^\alpha \sum_{k=0}^{N(y)} \frac{1}{(2^{k+1}\rho(y))^\alpha} \int_{2^{k+1}\rho(y) > d_x^\epsilon(v, v_0) > 2^k \rho(y)} f(v) d\mu_x(v) \\
&\leq 2^\alpha c_m^\epsilon \sum_{k=0}^{N(y)} \frac{1}{\mu_x(B(v_0, 2^{k+1}\rho(y)))^\epsilon} \int_{2^{k+1}\rho(y) > d_x^\epsilon(v, v_0)} f(v) d\mu_x(v) \\
&\leq 2^\alpha c_m^\epsilon \sum_{k=0}^{N(y)} \frac{1}{\mu_x(B(v_0, 2^{k+1}\rho(y)))} \int_{2^{k+1}\rho(y) > d_x^\epsilon(v, v_0)} f(v) d\mu_x(v) \\
&\leq 2^\alpha c_m^\epsilon \times N(y) \times \mathcal{M}f(v_0).
\end{aligned}$$

Set $R_0 := \max\{R, R'\}$. It follows from the definition of $N(y)$ that $y \in GH(\Lambda_\Gamma) \setminus B_X(x, R_0)$ implies

$$(3.6) \quad \frac{1}{C_q} \int_{\partial X \setminus B_\rho} f(v) \frac{P(x, y, v)^{\frac{\alpha}{2}}}{\varphi_x(y)} d\mu_x(v) \leq \left(\frac{2^{2\alpha} e^{\alpha(2\delta+c)} c_m^\alpha c'}{a} \right) \mathcal{M}f(v_0).$$

Inequality (3.4) combined with Inequality (3.6) give a constant C_0 to complete the proof. \square

Proof of Theorem 1.5. We follow the proof of [2, Théorème 7.7, p. 172]. Fix x in X and let f in the Banach space $L^1(\partial X, \mu_x)$ endowed with its standard norm $\|\cdot\|_1$. Define for all y in X :

$$(3.7) \quad \mathcal{L}_y f : v \in \Lambda_\Gamma \mapsto |(\overline{\mathcal{P}_0} f)(y) - f(v)| \in \mathbb{R}^+$$

and for $C > 0$, define

$$(3.8) \quad \mathcal{L}_C f : v \in \Lambda_\Gamma \mapsto \limsup_{\substack{y \rightarrow v \\ y \in \Omega_C(v)}} (\mathcal{L}_y f)(v) \in \mathbb{R}^+.$$

We shall prove that $\mathcal{L}_C f = 0$ almost everywhere with respect to μ_x . Let $\epsilon > 0$ and pick a continuous function g in $C(\partial X)$ such that $\|f - g\|_1 \leq \epsilon$. Observe that Proposition 3.1 implies that $\mathcal{L}_C g = 0$ and observe also that for all $C > 0$ and for $R > 0$:

$$\mathcal{L}_C f \leq \mathcal{N}_C^R f + |f|.$$

We have

$$\begin{aligned}
\mathcal{L}_C f &= \mathcal{L}_C(f - g) + \mathcal{L}_C g \\
&= \mathcal{L}_C(f - g) \\
&\leq \mathcal{N}_C^{R_0}(f - g) \\
&\leq C_0 \mathcal{M}(f - g) + |f - g|,
\end{aligned}$$

where the two positive constants R_0 and C_0 are yielded by Proposition 3.6 and the last inequality follows from Proposition 3.6. The maximal inequality of Proposition 2.7 implies that for all $s > 0$, we have

$$\mu_x(\{\mathcal{L}_C f > s\}) \leq 3^\alpha k C_0 \frac{\|f - g\|_1}{s} + \frac{\|f - g\|_1}{s}.$$

Since $\|f - g\|_1 \leq \epsilon$, the above inequality implies that for all $s > 0$ we have

$$\mu_x(\{\mathcal{L}_C f > s\}) = 0.$$

Hence,

$$\mathcal{L}_C f = 0$$

almost everywhere for all $C > 0$. To conclude, define $A_n := \{\mathcal{L}_n f = 0\}$ for $n \in \mathbb{N}^*$. We have shown that A_n is a set of full measure with respect to μ_x and thus $\cap_{n \geq 1} A_n$ is also a set of full measure. Hence the proof is complete. \square

4. APPLICATION TO DECAY OF MATRIX COEFFICIENTS OF BOUNDARY REPRESENTATIONS

4.1. The weak inequality of Harish-Chandra. Let Γ be a convex cocompact group of isometries of a proper CAT(-1) space X and let μ be a Γ -invariant conformal density of dimension $\alpha(\Gamma)$.

Consider a matrix coefficient associated with the boundary representation given by two essentially bounded functions $f, g \in L^\infty(\partial X)$, namely $\langle \pi_x(\cdot)f, g \rangle$ and observe that $|\langle \pi_x(\gamma)f, g \rangle| \leq \|f\|_\infty \|g\|_\infty \phi_x(\gamma)$ for all $\gamma \in \Gamma$. In other words, the matrix coefficient $\langle \pi_x(\cdot)f, g \rangle$ satisfies the weak inequality of Harish-Chandra of degree 0. Nevertheless, for all functions ξ which are not in L^∞ the matrix coefficient $\langle \pi_x(\cdot)\mathbf{1}_{\partial X}, \xi \rangle$ can not satisfy the weak inequality of degree 0 anymore. The proof is a direct application of theorem à la Fatou:

Proof of Corollary 1.8. Let x be in $GH(\Lambda_\Gamma)$, let ξ be in $L^2(\partial X, \mu_x)$ and assume that the Harish-Chandra's weak inequality of degree 0 holds: there exists $C > 0$ such that for all γ in Γ we have

$$\frac{\langle \pi_x(\gamma)\mathbf{1}_{\partial X}, \xi \rangle}{\phi_x(\gamma)} \leq C.$$

Observe that for all γ in Γ

$$(\mathcal{P}_0 \xi)(\gamma x) = \frac{\langle \pi_x(\gamma)\mathbf{1}_{\partial X}, \xi \rangle}{\phi_x(\gamma)}.$$

For all v is in Λ_Γ^{rad} , we know by Lemma 2.4 that there exist $C' > 0$ and a sequence $(\gamma_n)_{n \in \mathbb{N}} \in \Gamma$ such that $\gamma_n x \in \Omega_{C'}(v)$ for all $n \in \mathbb{N}$. Thus Theorem 1.5 implies that ξ is in $L^\infty(\partial X)$. Hence the weak inequality of Harish-Chandra of degree one must fail if ξ is not in $L^\infty(\partial X)$. \square

Remark 4.1. Let Γ be a non-uniform lattice (in rank one) acting on its symmetric space endowed with its Riemannian left-invariant metric and consider its boundary representation associated with the Lebesgue measure class. Then it cannot satisfy the weak inequality of Harish-Chandra of any degree. Because if the weak inequality holds, then it would follow that the boundary representation satisfies property RD and the presence of one parabolic element is an obstruction (see for example [7, Corollary 1.3]). Nevertheless, as we will see in Proposition 1.10 and 1.11, the decay of matrix coefficients is related to the property RD (see Remark 1.12).

4.2. Some results about the decay of matrix coefficients of boundary representations. Let Γ be in \mathcal{C} and consider a Γ -invariant quasi-conformal density of dimension $\alpha(\Gamma)$. Fix an origin x in X and denote $C_{n,\rho}(x)$ by $C_{n,\rho}$.

4.2.1. Using a uniform boundedness. Following the ideas of [6], to prove the irreducibility of boundary representations in [8] and in [15], it is proved that a sequence of functions is uniformly bounded with respect to the L^∞ norm. Recall the definition of such a sequence of functions $(F_{n,\rho})_{n > \rho}$:

$$(4.1) \quad F_{n,\rho} : v \in \partial X \mapsto \frac{1}{|C_{n,\rho}|} \sum_{\gamma \in C_{n,\rho}} \frac{\pi_x(\gamma)\mathbf{1}_{\partial X}}{\phi_x(\gamma)}(v) \in \mathbb{R}^+.$$

where ρ is a positive constant.

Proposition 4.2. If Γ is in \mathcal{C} , then there exists $\rho > 0$, $N_\rho \in \mathbb{N}$ and a positive constant $M > 0$ such that

$$\|F_{n,\rho}\|_\infty \leq M$$

for all $n > N_\rho$.

Indeed, we shall use the dual inequality.

Proposition 4.3. *If Γ is in \mathcal{C} , then there exists $\rho > 0$, $N_\rho \in \mathbb{N}$ and a positive constant $M > 0$ such that for all $f \in L^1(\partial X, \mu_x)$ we have*

$$\left| \frac{1}{|C_{n,\rho}|} \sum_{\gamma \in C_{n,\rho}} \mathcal{P}_0(f) \right| \leq M \|f\|_1$$

for all $n > N_\rho$.

Proof. The map

$$F \in L^\infty \mapsto \ell_F \in (L^1)^*$$

where $\ell_F(f) = \langle f, F \rangle$, is an isomorphism of Banach spaces, namely

$$\sup_{f \in L^1} \frac{|\ell_F(f)|}{\|f\|_1} = \|F\|_\infty.$$

Take the dual inequality of Proposition 4.2 via the Banach space isomorphism $\ell_{F_{n,\rho}}$ to obtain the result of Proposition 4.3. \square

In Proposition 4.2 and in Proposition 4.3 we may assume that $\rho > 1$. Indeed, we may choose all $\rho \geq \rho_0$ where ρ_0 guarantees that $|C_{n,\rho_0}| > 0$ for n large enough.

We need the following lemma:

Lemma 4.4. *Let $\mu = (\mu_x)_{x \in X}$ be a Γ -equivariant quasi-conformal density of dimension α , where Γ is a discrete group of isometries of X . Let x be in X and let ξ, η be in $L^2(\partial X, \mu_x)$ be two positive functions and let γ be in Γ . Then*

$$\frac{\langle \pi_x(\gamma)\xi, \eta \rangle^2}{\phi_x(\gamma)^2} \leq (\mathcal{P}_0 \xi^2)(\gamma^{-1}x)(\mathcal{P}_0 \eta^2)(\gamma x).$$

Proof. The Cauchy-Schwarz inequality implies :

$$\begin{aligned} \langle \pi_x(\gamma)\xi, \eta \rangle^2 &= \left(\int_{\partial X} \xi(\gamma^{-1}v) P(x, \gamma x, v)^{\frac{\alpha}{2}} \eta(v) d\mu_x(v) \right)^2 \\ &\leq \left(\int_{\partial X} \xi(\gamma^{-1}v)^2 P(x, \gamma x, v)^{\frac{\alpha}{2}} d\mu_x(v) \right) \left(\int_{\partial X} P(x, \gamma x, v)^{\frac{\alpha}{2}} \eta(v)^2 d\mu_x(v) \right) \\ &= \langle \pi_x(\gamma)\xi^2, 1_{\partial X} \rangle \langle \pi_x(\gamma)1_{\partial X}, \eta^2 \rangle \\ &= \langle \pi_x(\gamma^{-1})1_{\partial X}, \xi^2 \rangle \langle \pi_x(\gamma)1_{\partial X}, \eta^2 \rangle. \end{aligned}$$

Divide this inequality by $\phi_x(\gamma)^2$ and use Definition (1.1) to complete the proof. \square

We can prove Proposition 1.10:

Proof of Proposition 1.10. It is sufficient to prove that there exists a polynomial Q of degree 2 such that for all positive ξ and for all sufficiently large integers k

$$(4.2) \quad \sum_{C_{k,\rho}} \langle \pi_x(\gamma)1_{\partial X}, \xi \rangle^2 \leq Q(k) \|\xi\|_2^2.$$

Recall that there exists $R > 0$ such that the Harish-Chandra estimates hold on $GH(\Lambda_\Gamma) \setminus B_X(x, R)$. Thus all integers k such that $k \geq R + \rho$ we have

$$\begin{aligned}
\sum_{C_{k,\rho}} \langle \pi_x(\gamma) \mathbf{1}_{\partial X}, \xi \rangle^2 &= \sum_{C_{k,\rho}} \phi_x^2(\gamma) \frac{\langle \pi_x(\gamma) \mathbf{1}_{\partial X}, \xi \rangle^2}{\phi_x^2(\gamma)} \\
(\text{upper bound of the Harish-Chandra estimates}) &\leq Q_2^2(k) e^{-\alpha(\Gamma)k} \sum_{C_{k,\rho}} \frac{\langle \pi_x(\gamma) \mathbf{1}_{\partial X}, \xi \rangle^2}{\phi_x^2(\gamma)} \\
(\text{Lemma 4.4}) &\leq Q_2^2(k) e^{-\alpha(\Gamma)k} \sum_{C_{k,\rho}} \frac{\langle \pi_x(\gamma) \mathbf{1}_{\partial X}, \xi^2 \rangle}{\phi_x(\gamma)} \\
&= Q_2^2(k) e^{-\alpha(\Gamma)k} \sum_{C_{k,\rho}} \mathcal{P}_0(\xi^2) \\
&\leq C' Q_2^2(k) \frac{1}{|C_{k,\rho}|} \sum_{C_{k,\rho}} \mathcal{P}_0(\xi^2)
\end{aligned}$$

The last inequality follows from the fact that for convex cocompact groups, hyperbolic groups and lattices in rank one, we can find $C' > 0$ such that

$$e^{-\alpha(\Gamma)k} \leq C' \frac{1}{|C_{k,\rho}|}.$$

Apply Proposition 4.3 to the last inequality of the above computation to obtain Inequality (4.2) with a polynomial Q of degree 2. This finishes the proof. \square

4.2.2. With a naive application of a theorem of T. Roblin. Recall the equidistribution theorem that we shall use and that we have already used in the paper [8]. We refer to [8, Preliminaires] for the definition of arithmetic spectrum and for the definition of Bowen-Margulis-Sullivan measure. Notice that convex cocompact groups with non-arithmetic spectrum and lattices in rank one semisimple Lie groups satisfy the hypothesis of the following theorem.

Let D_z be the unit Dirac mass at z and let m_Γ be the Bowen-Margulis-Sullivan measure.

Theorem 4.5. (*T. Roblin*) *Let Γ be a discrete group of isometries of X with a non-arithmetic spectrum. Assume that Γ admits a finite Bowen-Margulis-Sullivan measure associated with a Γ -invariant conformal density μ of dimension $\alpha = \alpha(\Gamma)$. Then, for all $x, y \in X$ we have:*

$$\alpha e^{-\alpha n} \|m_\Gamma\| \sum_{\{\gamma \in \Gamma \mid d(x, \gamma y) < n\}} D_{\gamma^{-1}x} \otimes D_{\gamma y} \rightharpoonup \mu_x \otimes \mu_y$$

as $n \rightarrow +\infty$ with respect to the weak* convergence of $C(\overline{X} \times \overline{X})^*$.

Proof of Proposition 1.11. Assume first that x is in $GH(\Lambda_\Gamma)$, let $\rho > 0$ and let f and g in $L^2(\partial X, \mu_x)$. Recall that $C_{n,\rho}(x) = \{n - \rho \leq |\gamma|_x < n + \rho\}$. Lemma 4.4 implies

$$|\langle \pi_x(\gamma) f, g \rangle|^2 \leq \varphi_x(\gamma)^2 (D_{\gamma^{-1}x} \otimes D_{\gamma x})(\mathcal{P}_0 f^2 \otimes \mathcal{P}_0 g^2).$$

If γ is in $C_{n,\rho}(x)$, the upper bound of the Harish-Chandra estimates (see Proposition 2.5) implies that there exists Q_2 such that

$$\varphi_x(\gamma)^2 \leq Q_2(n) e^{-\alpha(\Gamma)(n+\rho)}.$$

It follows that for n large enough we have:

$$\sum_{\gamma \in C_{n,\rho}(x)} |\langle \pi_x(\gamma) f, g \rangle|^2 \leq Q_2(n) e^{-\alpha(\Gamma)(n+\rho)} \sum_{\gamma \in C_{n,\rho}(x)} (D_{\gamma^{-1}x} \otimes D_{\gamma x})(\mathcal{P}_0 f^2 \otimes \mathcal{P}_0 g^2).$$

If f and g are continuous functions, the lower bound of Harish-Chandra estimates combined with Proposition 3.1 imply that $\overline{\mathcal{P}_0 f^2} \otimes \overline{\mathcal{P}_0 g^2}$ is a continuous function on $\overline{X} \times \overline{X}$. Set

$$Q = \frac{1}{\alpha(\Gamma) \|m_\Gamma\|} Q_2.$$

Since $C_{n,\rho}(x) \subset \Gamma_{n+\rho}(x)$, we have for n large enough:

$$\frac{1}{Q(n)} \sum_{\gamma \in C_{n,\rho}(x)} |\langle \pi_x(\gamma)f, g \rangle|^2 \leq \|m_\Gamma\| \alpha(\Gamma) e^{-\alpha(\Gamma)(n+\rho)} \sum_{\gamma \in \Gamma_{n+\rho}(x)} (D_{\gamma^{-1}x} \otimes D_{\gamma x})(\overline{\mathcal{P}_0} f^2 \otimes \overline{\mathcal{P}_0} g^2).$$

Theorem 4.5 implies for all $\rho > 0$:

$$\begin{aligned} \limsup_{n \rightarrow +\infty} \frac{1}{Q(n)} \sum_{\gamma \in C_{n,\rho}(x)} |\langle \pi_x(\gamma)f, g \rangle|^2 &\leq \lim_{n \rightarrow +\infty} \|m_\Gamma\| \alpha(\Gamma) e^{-\alpha(\Gamma)(n+\rho)} \sum_{\gamma \in \Gamma_{n+\rho}(x)} (D_{\gamma^{-1}x} \otimes D_{\gamma x})(\overline{\mathcal{P}_0} f^2 \otimes \overline{\mathcal{P}_0} g^2) \\ &= (\mu_x \otimes \mu_x)(\overline{\mathcal{P}_0} f^2 \otimes \overline{\mathcal{P}_0} g^2) \\ &= \mu_x(\overline{\mathcal{P}_0} f^2) \mu_x(\overline{\mathcal{P}_0} g^2) \\ &= \mu_x(f^2) \mu_x(g^2) \\ &= \|f\|_2^2 \|g\|_2^2. \end{aligned}$$

Hence we have proved for all x in $GH(\Lambda_\Gamma)$, for all $\rho > 0$, there exists Q such that

$$(4.3) \quad \limsup_{n \rightarrow +\infty} \frac{1}{Q(n)} \sum_{\gamma \in C_{n,\rho}(x)} |\langle \pi_x(\gamma)f, g \rangle|^2 \leq \|f\|_2^2 \|g\|_2^2.$$

If x is not in $GH(\Lambda_\Gamma)$, pick x' in $GH(\Lambda_\Gamma)$ and observe that

$$(4.4) \quad C_{n,\rho}(x) \subset C_{n,\rho+2d(x,x')}(x').$$

Then, define the multiplication operator $M_{x,x'}$ defined as

$$(4.5) \quad M_{x,x'} : \xi \in L^2(\partial X, \mu_x) \mapsto m_{x,x'} \xi \in L^2(\partial X, \mu_{x'}),$$

where

$$m_{x,x'} : v \in \partial X \mapsto e^{\frac{\alpha}{2}\beta_v(x,x')},$$

which is an isometry and which intertwines π_x and $\pi_{x'}$, namely

$$\pi_{x'} M_{x,x'} = M_{x,x'} \pi_x.$$

Fix $\rho > 0$ and use the operator $M_{x,x'}$ to apply Inequality (4.3) with x' , $\rho + 2d(x, x')$ to complete the proof. □

5. HARISH-CHANDRA'S SCHWARTZ ALGEBRA OF DISCRETE GROUPS

5.1. Results. Recall that the space $C_c(\Gamma)$ is an algebra for the convolution product defined as

$$f_1 * f_2(g) = \sum_{\gamma \in \Gamma} f_1(\gamma) f_2(\gamma^{-1}g), \quad \forall g \in \Gamma.$$

It is natural to ask if the Harish-Chandra's Schwartz space $\mathcal{S}_t(\Gamma)$, for t large enough, has a structure of a convolution algebra. The answer is positive. The only difficulty was to find suitable conditions which play analogous roles of the conditions (HC3c) and (HC3d) in [19, Chapitre 4, p. 82]. We replace the averaging process over the compact K in the context of semisimple Lie groups by sums over annuli, based on Proposition 4.3.

Fix x an origin in X and denote $d(x, \gamma x)$ by $|\gamma|$, $\{\gamma \in \Gamma \text{ such that } |\gamma| \leq n\}$ by Γ_n and $\Gamma_n \setminus \Gamma_{n-1}$ by C_n .

5.2. An inequality for the Harish-Chandra function.

Lemma 5.1. *Let Γ be in \mathcal{C} . There exists t_0 such that for all $t \geq t_0$ there exists C_t satisfying for all g in Γ :*

$$\sum_{\gamma \in \Gamma} \frac{\phi_x(g\gamma^{-1})\phi_x(\gamma)}{(1+|\gamma|)^t} \leq C_t \phi_x(g).$$

Proof. Let g be in Γ and observe that for all $t > 0$ and for all integer N we have:

$$(5.1) \quad \sum_{\gamma \in \Gamma} \frac{\phi_x(g\gamma^{-1})\phi_x(\gamma)}{(1+|\gamma|)^t} = \sum_{\gamma \in \Gamma_N} \frac{\phi_x(g\gamma^{-1})\phi_x(\gamma)}{(1+|\gamma|)^t} + \sum_{n=N+1}^{\infty} \sum_{\gamma \in C_n} \frac{\phi_x(g\gamma^{-1})\phi_x(\gamma)}{(1+|\gamma|)^t}.$$

Consider the first term: observe that Cauchy-Schwarz inequality implies that for all $\gamma \in \Gamma$ we have $\phi_x(\gamma) \leq 1$. Then observe also that $\phi_x(g\gamma^{-1}) = \langle \pi_x(g)(\pi(\gamma^{-1})\mathbf{1}_{\partial X}), \mathbf{1}_{\partial X} \rangle$. Since the function $\pi(\gamma^{-1})\mathbf{1}_{\partial X}$ is a positive continuous function on a compact set and since the representation π_x preserves the cone of positive function we have

$$\phi_x(g\gamma^{-1}) \leq C_\gamma \phi_x(g),$$

where $C_\gamma := \sup\{\pi(\gamma^{-1})\mathbf{1}_{\partial X}(v), v \in \partial X\}$ is some positive constant. Hence for all $t > 0$ and for all N there exists a positive constant C' such that

$$(5.2) \quad \sum_{\gamma \in \Gamma_N} \frac{\phi_x(g\gamma^{-1})\phi_x(\gamma)}{(1+|\gamma|)^t} \leq C' \phi_x(g).$$

Consider now the second term:

Since Γ is in \mathcal{C} recall that for all $\rho > 0$ there exists a polynomial Q such that for all integers n large enough and for all $\gamma \in C_{n,\rho}$ we have:

$$(5.3) \quad \phi_x(\gamma) \leq Q(n) \frac{1}{|C_{n,\rho}|}.$$

Let $\rho > 1$ such that $C_n \subset C_{n,\rho}$. Using Inequality (5.3) we have for all $n \geq \rho$:

$$\begin{aligned} \sum_{\gamma \in C_n} \phi_x(g\gamma^{-1})\phi_x(\gamma) &= \sum_{\gamma \in C_n} \phi_x(\gamma)^2 \frac{\phi_x(g\gamma^{-1})}{\phi_x(\gamma)} \\ &\leq \frac{Q(n)}{|C_{n,\rho}|} \sum_{\gamma \in C_n} \frac{\phi_x(g\gamma^{-1})}{\phi_x(\gamma)} \\ &= \frac{Q(n)}{|C_{n,\rho}|} \sum_{\gamma \in C_{n,\rho}} \frac{\langle \pi_x(g\gamma^{-1})\mathbf{1}, \mathbf{1} \rangle}{\phi_x(\gamma)} \\ &= \frac{Q(n)}{|C_{n,\rho}|} \sum_{\gamma \in C_{n,\rho}} \frac{\langle \pi_x(\gamma)\mathbf{1}, \pi_x(g^{-1})\mathbf{1} \rangle}{\phi_x(\gamma)} \\ &= \frac{Q(n)}{|C_{n,\rho}|} \sum_{\gamma \in C_{n,\rho}} D_{\gamma x}(\mathcal{P}_0 f) \end{aligned}$$

where

$$f = \pi_x(g^{-1})\mathbf{1} \in L^1(\partial X, \mu_x).$$

Observe that Equality (1.7) implies

$$\|\pi_x(g^{-1})\mathbf{1}\|_1 = \phi_x(g^{-1}) = \phi_x(g).$$

Applying Proposition 4.3 to f we have for n large enough:

$$\sum_{\gamma \in C_n} \phi_x(g\gamma^{-1})\phi_x(\gamma) \leq Q(n)\phi_x(g).$$

Thus there exists t_0 such that for all $t \geq t_0$ there exists a constant $C_t > 0$ satisfying for all $g \in \Gamma$:

$$\sum_{\gamma \in \Gamma} \frac{\phi_x(g\gamma^{-1})\phi_x(\gamma)}{(1+|\gamma|)^t} \leq C_t \phi_x(g).$$

□

5.3. Convolution algebra.

Proposition 5.2. *Let Γ be in \mathcal{C} . There exists $t_0 > 0$, such that for all $t \geq t_0$ we have $\mathcal{S}_t(\Gamma) * \mathcal{S}_t(\Gamma) \subset \mathcal{S}_t(\Gamma)$.*

Proof. We follow the proof of V. Lafforgue in [19, Proposition 4.3]. Let $t > 0$ and let f_1, f_2 be in $\mathcal{S}_t(\Gamma)$. We have for all $\gamma \in \Gamma$

$$|f_i(\gamma)| \leq C_i \frac{\phi_x(\gamma)}{(1+|\gamma|)^t}$$

for some $C_i > 0$ with $i = 1, 2$. Thus for all $g \in \Gamma$ we have

$$\begin{aligned} \frac{1}{C_1 C_2} |f_1 * f_2(g)| &= \frac{1}{C_1 C_2} \sum_{\gamma \in \Gamma} f_1(\gamma) f_2(\gamma^{-1}g) \\ &\leq \sum_{\gamma \in \Gamma} \frac{\phi_x(\gamma)}{(1+|\gamma|)^t} \frac{\phi_x(\gamma^{-1}g)}{(1+|\gamma^{-1}g|)^t} \\ &= \sum_{\{|\gamma| \leq \frac{|g|}{2}\}} \frac{\phi_x(\gamma)}{(1+|\gamma|)^t} \frac{\phi_x(\gamma^{-1}g)}{(1+|\gamma^{-1}g|)^t} + \sum_{\{|\gamma| > \frac{|g|}{2}\}} \frac{\phi_x(\gamma)}{(1+|\gamma|)^t} \frac{\phi_x(\gamma^{-1}g)}{(1+|\gamma^{-1}g|)^t}. \end{aligned}$$

Observe that $|\gamma| \leq \frac{|g|}{2} \Rightarrow |\gamma^{-1}g| \geq \frac{|g|}{2}$.

We have for the first sum:

$$\begin{aligned} \sum_{\{|\gamma| \leq \frac{|g|}{2}\}} \frac{\phi_x(\gamma)}{(1+|\gamma|)^t} \frac{\phi_x(\gamma^{-1}g)}{(1+|\gamma^{-1}g|)^t} &\leq \frac{2^t}{(1+|g|)^t} \sum_{\{|\gamma| \leq \frac{|g|}{2}\}} \frac{\phi_x(\gamma)\phi_x(\gamma^{-1}g)}{(1+|\gamma|)^t} \\ &\leq \frac{2^t}{(1+|g|)^t} \sum_{\gamma \in \Gamma} \frac{\phi_x(\gamma)\phi_x(\gamma^{-1}g)}{(1+|\gamma|)^t}. \end{aligned}$$

We have for the second sum:

$$\begin{aligned} \sum_{\{|\gamma| > \frac{|g|}{2}\}} \frac{\phi_x(\gamma)}{(1+|\gamma|)^t} \frac{\phi_x(\gamma^{-1}g)}{(1+|\gamma^{-1}g|)^t} &\leq \frac{2^t}{(1+|g|)^t} \sum_{\{|\gamma| > \frac{|g|}{2}\}} \frac{\phi_x(\gamma)\phi_x(\gamma^{-1}g)}{(1+|\gamma^{-1}g|)^t} \\ &\leq \frac{2^t}{(1+|g|)^t} \sum_{\gamma \in \Gamma} \frac{\phi_x(\gamma)\phi_x(\gamma^{-1}g)}{(1+|\gamma^{-1}g|)^t} \\ &= \frac{2^t}{(1+|g|)^t} \sum_{\gamma \in \Gamma} \frac{\phi_x(g\gamma^{-1})\phi_x(\gamma)}{(1+|\gamma|)^t} \end{aligned}$$

where the last inequality comes from the change of variable $\gamma' = \gamma^{-1}g$.

Lemma 5.1 implies that the two terms are bounded by $\phi_x(g)$ for all $t \geq t_0$ for some positive real number t_0 . Hence, there exists t_0 such that for all $t \geq t_0$ we have for all $g \in \Gamma$:

$$f_1 * f_2(g) \leq B_t \frac{\phi_x(g)}{(1+|g|)^t},$$

for some positive constant B_t . □

Proposition 5.3. *There exists $t_0 > 0$, such that for all $t \geq t_0$ we have $\mathcal{S}_t(\Gamma) * \ell^2(\Gamma) \subset \ell^2(\Gamma)$.*

Proof. We follow the proof of V. Lafforgue in [19, Proposition 5.2]. Let h be in $\ell^2(\Gamma)$. It is enough to prove that for all t large enough there exists a constant $C_t > 0$ such that

$$\sum_{g \in \Gamma} \left(\sum_{\gamma \in \Gamma} \frac{\phi_x(\gamma)}{(1 + |\gamma|)^t} h(\gamma^{-1}g) \right)^2 \leq C_t \|h\|_2^2.$$

First, we use Cauchy-Schwarz inequality to obtain:

$$(5.4) \quad \left(\sum_{\gamma \in \Gamma} \frac{\phi_x(\gamma)}{(1 + |\gamma|)^t} h(\gamma^{-1}g) \right)^2 \leq \left(\sum_{\gamma \in \Gamma} \frac{\phi_x(\gamma)}{(1 + |\gamma|)^t} \frac{h^2(\gamma^{-1}g)}{\phi_x(\gamma^{-1}g)} \right) \left(\sum_{\gamma \in \Gamma} \frac{\phi_x(\gamma^{-1}g)\phi_x(\gamma)}{(1 + |\gamma|)^t} \right).$$

Lemma 5.1 implies that there exists $t_0 > 0$ such that for all $t \geq t_0$ there exists C_t satisfying for all $g \in \Gamma$:

$$(5.5) \quad \sum_{\gamma \in \Gamma} \frac{\phi_x(\gamma^{-1}g)\phi_x(\gamma)}{(1 + |\gamma|)^t} \leq C_t \phi_x(g).$$

Combining Inequality (5.4) with Inequality (5.5) we obtain for all $t \geq t_0$:

$$\begin{aligned} \sum_{g \in \Gamma} \left(\sum_{\gamma \in \Gamma} \frac{\phi_x(\gamma)}{(1 + |\gamma|)^t} h(\gamma^{-1}g) \right)^2 &\leq C_t \sum_{(\gamma, g) \in \Gamma \times \Gamma} \frac{\phi_x(\gamma)}{(1 + |\gamma|)^t} \frac{h^2(\gamma^{-1}g)}{\phi_x(\gamma^{-1}g)} \phi_x(g) \\ &= C_t \sum_{(\gamma, \gamma') \in \Gamma \times \Gamma} \frac{\phi_x(\gamma)}{(1 + |\gamma|)^t} \frac{h^2(\gamma')}{\phi_x(\gamma')} \phi_x(\gamma\gamma') \\ &= C_t \sum_{\gamma' \in \Gamma} \left(\sum_{\gamma \in \Gamma} \frac{\phi_x(\gamma)\phi_x(\gamma\gamma')}{(1 + |\gamma|)^t} \right) \frac{h^2(\gamma')}{\phi_x(\gamma')} \\ &\leq C_t^2 \|h\|_2^2 < +\infty, \end{aligned}$$

where the last equality comes from a slight modification of Lemma 5.1 using Equality (1.7). The proof is complete. \square

Remark 5.4. Proposition 5.3 proves that we have the following representation of $\mathcal{S}_t(\Gamma)$ for t large enough:

$$\begin{aligned} \mathcal{S}_t(\Gamma) &\rightarrow B(\ell^2(\Gamma)) \\ f &\mapsto M_f \end{aligned}$$

where $M_f \xi = f * \xi$ for all $\xi \in \ell^2(\Gamma)$ and $\|M_f \xi\|_2 \leq C \|f\|_{\mathcal{S}_t(\Gamma)} \|\xi\|_2$ for some positive constant C depending on t . It follows that

$$\|M_f\|_{op} \leq C \|f\|_{\mathcal{S}_t(\Gamma)},$$

where $\|\cdot\|_{op}$ denotes the operator norm.

The proof of Proposition 1.14 is almost done:

Proof. Proposition 5.2 proves that $\mathcal{S}_t(\Gamma)$ has the structure of a convolution algebra for t large enough. Proposition 5.3 shows that $\mathcal{S}_t(\Gamma)$ for t large enough is represented in the algebra of bounded operators on $\ell^2(\Gamma)$. Since $C_c(\Gamma) \subset \mathcal{S}_t(\Gamma)$ for all $t > 0$ and thanks to Remark 5.4 it is clear that $\mathcal{S}_t(\Gamma)$, for t large enough, generates on $\ell^2(\Gamma)$ the reduced C^* -algebra and the von Neumann algebra of Γ . \square

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